Problem Set 2 – Phys 4510 Optics – Fall 2014

Due date: Tu, September 16, in class

Reading: Hecht 3.4.3, 3.4.4, 3.5

1. In this problem we address the fundamental differences between the response of materials at optical frequencies compared to low-frequencies and DC.

   a) The DC and low-frequency relative dielectric permittivity of water varies from 88 at 0 °C to 55 at 100 °C. Explain this behavior.

   b) Explain why compared to water the DC dielectric permittivity of ice is about 80 times smaller. Therefore, why ice is nearly transparent to radar beams, yet considerably perturbed by rain?

   c) Over the same temperature range as above, the index of refraction for light at 600 nm varies only from about 1.33 to 1.32. Why is the change of n so much smaller at optical frequencies than the change in ϵ_r at low frequencies?

2. The resonant frequency of electronic absorption of lead glass is in the UV fairly near the visible, in contrast to fused silica or quartz, where the resonance is far into the UV. Make a qualitative sketch of $n(\lambda)$ for both glasses and explain which one has a higher n in the visible spectral region.

3. Augustin Louis Cauchy (1789 - 1857) determined an empirical equation around 1836 to describe the wavelength dependence of transparent media, in the form of a power series. This Cauchy equation, even in leading order provides a reasonably good approximation in many cases. It is still often in use today and is given by:

$$n(\lambda) = B + \frac{C}{\lambda^2}. \tag{1}$$

   with the empirical coefficients B and C. (Note that λ refers to the vacuum wavelength). For borosilicate glass (BK7, a glass often used for lenses and other optical components for the visible spectral range), $B = 1.5046$ and $C = 0.00420 \mu m^2$. Use that expression to calculate phase and group velocity for $\lambda = 400$, 532, 632, and 800 nm, and discuss the trends you find.

4. Later on, in 1871 Wilhelm Sellmeier, still without knowing much yet about microscopic resonant processes in atoms, molecules, or solids, proposed an alternative equation to describe the wavelength dependence of the index of refraction of the form:

$$n(\lambda_0)^2 = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_{0,j}^2}, \tag{2}$$

   with $\lambda \gg \lambda_{0,j}$. Show that the Cauchy equation in the general form $n(\lambda) = C_1 + C_2/\lambda^2 + C_3/\lambda^3 + ...$ is an approximation of the more general Sellmeier equation. Note how the Sellmeier equation already has a signature of the Lorentz model with a resonance denominator, although not yet of the harmonic oscillator form.

5. Derive the formulas:

$$v_g = v_\phi - \lambda \frac{dv_\phi}{d\lambda} \tag{3}$$

   and

$$\frac{1}{v_g} = \frac{1}{v_\phi} - \frac{\lambda_0}{c} \frac{dn}{d\lambda_0} \tag{4}$$

6. Determine the relative scattering intensity of a dilute gas for $\lambda = 580$ nm and $\lambda = 400$ nm. Explain based on this result the color of the sun at sunset.

7. The driven and damped classical harmonic oscillator: I almost apologize for issuing the following as a homework in this class, because you have seen this many times before. However, to carry it out for the specific case of an optical dielectric response is instructive and we will often resort back to that result.

   The light matter interaction can classically be described by assuming a harmonic motion of an electron in the potential of the nucleus (electron on a spring). In 1D the equation of motion, including damping, then reads:

$$m\ddot{x} + m\Gamma\dot{x} + m\omega_0^2 x = eE_0 \mathrm{e}^{-i\omega t} \tag{5}$$

   The following steps will walk you through the solution. This is one approach. If – for good reasons or not, or from previous classes you know a different approach – show that instead, but either way go through the derivation carefully.

   a) First consider the case without driving field $\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = 0$. Making the Ansatz $x(t) = C\mathrm{e}^{\alpha t}$ derive the eigenvalues for the homogeneous differential equations (two solutions). The general solution is then the linear combinations of both terms.

   b) As you now see you have to distinguish different cases. Write the different solutions $x(t)$ and identify them as underdamped, critically damped, and overdamped case. Note for the underdamped case how the damping detunes the frequency. Plot for the underdamped case the solution for the polarization $p(t) = e\,x(t)$ for an IR lattice vibration response of a material with $\omega_0 = 900$ cm^{-1} and $\Gamma = 0.5, 5,$ and 20 cm^{-1} (note the units used: convert to time).

   c) Now we consider the inhomogeneous case, i.e., eq. (5): Apply the electric field for times much longer than the decay time so solutions are only of interest for times $t \gg 1/\Gamma \gg 1/\omega_0$ so any homogeneous solution that is initially present will have decayed away (we will expand this later to excitations that last shorter than the decay time). This makes it simple: a particular solution is $x(t) = x_0 \mathrm{e}^{-i\omega_0 t}$. Derive x_0.

   d) The induced optical polarization when the light interacts with a medium can be written as: $p(t) = nex(t) = \chi\epsilon_0 E(t)$ where χ is called the optical susceptibility of the medium. Using the result from (c), write the susceptibility χ and plot its real and imaginary part for the vibrational oscillator from above. Then use χ to derive an expression for the index of refraction n and extinction coefficient κ and again plot their frequency dependence.