

Solution Set 2 – Phys 4510 Optics – Fall 2014

Due date: Tu, September 16, in class

Scoring rubric

4 points/sub-problem, total: 40 points

3: Small mistake in calculation or formula

2: Correct formula but calculation error

1: Partial credit depend on the answer

Reading: Hecht 3.4.3, 3.4.4, 3.5

1. In this problem we address the fundamental differences between the response of materials at optical frequencies compared to low-frequencies and DC.

- a) The DC and low-frequency relative dielectric permittivity of water varies from 88 at 0 °C to 55 at 100 °C. Explain this behavior.

Solution:

Thermal agitation of the molecular dipoles causes a marked reduction in ϵ_r .

- b) Explain why compared to water the DC dielectric permittivity of ice is about 80 times smaller. Therefore, why ice is nearly transparent to radar beams, yet considerably perturbed by rain?

Solution:

The polar molecule, water, in the liquid state, is relatively free to move in response to the incident radiation. In the solid state, the molecules are not free to move. The radar (microwave) interacts strongly with the liquid droplets in rain.

- c) Over the same temperature range as above, the index of refraction for light at 600 nm varies only from about 1.33 to 1.32. Why is the change of n so much smaller at optical frequencies than the change in ϵ_r at low frequencies?

Solution:

Thermal agitation of the molecular dipoles has little effect on n . At optical frequencies n is predominantly due to electronic polarization, rotations of the molecular dipoles having ceased to be effective at much lower frequencies.

2. The resonant frequency of electronic absorption of lead glass is in the UV fairly near the visible, in contrast to fused silica or quartz, where the resonance is far into the UV. Make a qualitative sketch of $n(\lambda)$ for both glasses and explain which one has a higher n in the visible spectral region.

Solution:

We know the relation between refractive index and frequency (Eq 3.72 of Hecht, and also from the problem 7). With ω in the visible, $(\omega_0^2 - \omega^2)$ is smaller for lead glass and larger for fused silica or quartz. Hence $n(\omega)$ of lead glass is larger than fused silica or quartz. You can see qualitative sketch of $n(\lambda)$ in Figure 3.40 of Hecht.

3. Augustin Louis Cauchy (1789 - 1857) determined an empirical equation around 1836 to describe the wavelength dependence of transparent media, in the form of a power series. This Cauchy equation, even in leading order provides a reasonably good approximation in many cases. It is still often in use today and is given by:

$$n(\lambda) = B + \frac{C}{\lambda^2}. \quad (1)$$

with the empirical coefficients B and C . (Note that λ refers to the vacuum wavelength). For borosilicate glass (BK7, a glass often used for lenses and other optical components for the visible spectral range), $B = 1.5046$ and $C = 0.00420\mu\text{m}^2$. Use that expression to calculate phase and group velocity for $\lambda = 400, 532, 632,$ and 800 nm, and discuss the trends you find.

Solution:

Using the second equation of the problem 5 we can make the following table:

λ_0 [nm]	$n(\lambda_0)$	v_ϕ	v_g
400	1.531	$0.653c$	$0.632c$
532	1.519	$0.658c$	$0.646c$
632	1.515	$0.660c$	$0.651c$
800	1.511	$0.662c$	$0.656c$

From the equation for the refractive index we can see that for increasing λ the refractive index tends towards B . Thus, the group velocity tends towards the phase velocity for increasing wavelength. It will, however, never exceed the phase velocity.

4. Later on, in 1871 Wilhelm Sellmeier, still without knowing much yet about microscopic resonant processes in atoms, molecules, or solids, proposed an alternative equation to describe the wavelength dependence of the index of refraction of the form:

$$n(\lambda_0)^2 = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_{0,j}^2}, \quad (2)$$

with $\lambda \gg \lambda_{0,j}$. Show that the Cauchy equation in the general form $n(\lambda) = C_1 + C_2/\lambda^2 + C_3/\lambda^3 + \dots$ is an approximation of the more general Sellmeier equation. Note how the Sellmeier equation already has a signature of the Lorentz model with a resonance denominator, although not yet of the harmonic oscillator form.

Solution:

Binomially expanding $n^2 \sim 1 + A/(1 - \lambda_0^2/\lambda^2)$ gives $n^2 \sim 1 + A(1 + \lambda_0^2/\lambda^2)$ or $n^2 = (1 + A)[(1 + A\lambda_0^2/(1 + A)\lambda^2)]$. Taking the square root and expanding again gives $n \sim (1 + A)^{1/2}[1 + A\lambda_0^2/2(1 + A)\lambda^2]$. This has the Cauchy form with $C_1 = (1 + A)^{1/2}$ and $C_2 = A\lambda_0^2/2(1 + A)^{1/2}$.

5. Derive the formulas:

$$v_g = v_\phi - \lambda \frac{dv_\phi}{d\lambda} \quad (3)$$

and

$$\frac{1}{v_g} = \frac{1}{v_\phi} - \frac{\lambda_0}{c} \frac{dn}{d\lambda_0} \quad (4)$$

Solution:

$$v_g = \frac{d\omega}{dk} = \frac{d(v_\phi k)}{dk} = v_\phi + k \frac{dv_\phi}{dk} = v_\phi + k \frac{dv_\phi}{d\lambda} \frac{d\lambda}{dk}$$

Using the relation between $\lambda = \frac{2\pi}{k}$ we get for $\frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$ and thus for v_ϕ

$$v_g = v_\phi + k \frac{-2\pi}{k^2} \frac{dv_\phi}{d\lambda} = v_\phi - \frac{2\pi}{k} \frac{dv_\phi}{d\lambda} = v_\phi - \lambda \frac{dv_\phi}{d\lambda}$$

So we have proven the first relation. The second is just as straightforward.

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d}{d\omega} \left(\frac{n\omega}{c} \right) = \frac{n}{c} + \frac{\omega}{c} \frac{dn}{d\omega} = \frac{1}{v_\phi} + \frac{\omega}{c} \frac{dn}{d\lambda_0} \frac{d\lambda_0}{d\omega}$$

Now we have to be careful not to mix up the wavelength in vacuum λ_0 and wavelength in matter $\lambda = \frac{\lambda_0}{n}$. Using the relation between the wavelength and the wavenumber $\lambda_0 = \frac{2\pi}{k_0} = \frac{2\pi c}{\omega}$ we get $\frac{d\lambda_0}{\omega} = -\frac{2\pi c}{\omega^2}$. This yields

$$\frac{1}{v_g} = \frac{1}{v_\phi} + \frac{\omega}{c} \frac{dn}{d\lambda_0} \left(-\frac{2\pi c}{\omega^2} \right) = \frac{1}{v_\phi} - \frac{\lambda_0}{c} \frac{dn}{d\lambda_0}$$

6. Determine the relative scattering intensity of a dilute gas for $\lambda = 580$ nm and $\lambda = 400$ nm. Explain based on this result the color of the sun at sunset.

Solution:

Rayleigh scattering intensity is proportional to the $(1/\lambda)^4$. Hence

$$\left(\frac{1}{400}\right)^4 / \left(\frac{1}{580}\right)^4 = 4.42$$

At sunset, light must travel farther through the atmosphere before it gets to you, so more of it is reflected and scattered and the sun appears dimmer. The color of the sun itself appears to change, first to orange and then to red because even more of the short wavelength blues and greens are now scattered and only the longer wavelengths (reds, oranges) are left to be seen.

7. The driven and damped classical harmonic oscillator: I almost apologize for issuing the following as a homework in this class, because you have seen this many times before. However, to carry it out for the specific case of an optical dielectric response is instructive and we will often resort back to that result.

The light matter interaction can classically be described by assuming a harmonic motion of an electron in the potential of the nucleus (electron on a spring). In 1D the equation of motion, including damping, then reads:

$$m\ddot{x} + m\Gamma\dot{x} + m\omega_0^2 x = eE_0 e^{-i\omega t} \quad (5)$$

The following steps will walk you through the solution. This is one approach. If – for good reasons or not, or from previous classes you know a different approach – show that instead, but either way go through the derivation carefully.

- a) First consider the case without driving field $\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = 0$. Making the Ansatz $x(t) = Ce^{\alpha t}$ derive the eigenvalues for the homogeneous differential equations (two solutions). The general solution is then the linear combinations of both terms.

Solution:

First we calculate the time derivatives of the ansatz:

$$\begin{aligned}x(t) &= Ce^{\alpha t} \\ \dot{x}(t) &= C\alpha e^{\alpha t} \\ \ddot{x}(t) &= C\alpha^2 e^{\alpha t}\end{aligned}$$

From the undriven version of the differential equation we then get quickly by inserting the above derivatives

$$\alpha^2 + \Gamma\alpha + \omega_0^2 = 0$$

From there we can easily calculate the two expressions for α which will then give us the general expression for $x(t)$

$$\begin{aligned}\alpha_+ &= -\frac{\Gamma}{2} + \sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_0^2} \\ \alpha_- &= -\frac{\Gamma}{2} - \sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_0^2}\end{aligned}$$

The general solution of the undriven differential equation above is then

$$\begin{aligned}x(t) &= C_1 e^{\alpha_+ t} + C_2 e^{\alpha_- t} \\ x(t) &= e^{-\frac{\Gamma}{2}t} \left(C_1 e^{\sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_0^2}t} + C_2 e^{-\sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_0^2}t} \right)\end{aligned}$$

Of this general solution only the real part is physically relevant.

- b) As you now see you have to distinguish different cases. Write the different solutions $x(t)$ and identify them as underdamped, critically damped, and overdamped case. Note for the underdamped case how the damping detunes the frequency. Plot for the underdamped case the solution for the polarization $p(t) = \epsilon x(t)$ for an IR lattice vibration response of a material with $\omega_0 = 900 \text{ cm}^{-1}$ and $\Gamma = 0.5, 5,$ and 20 cm^{-1} (note the units used: convert to time).

Solution:

In the overdamped case

$$\left(\frac{\Gamma}{2}\right)^2 > \omega_0^2$$

The squareroot in the exponential is real and increases the time constant of the exponential decrease.

In the critically damped case

$$\left(\frac{\Gamma}{2}\right)^2 = \omega_0^2$$

the squareroot in the exponential vanishes and only the exponential decrease of the function remains.

In the underdamped case

$$\left(\frac{\Gamma}{2}\right)^2 < \omega_0^2$$

the squareroot in the exponential becomes negative and therefore forms an exponentially damped and oscillating solution.

The plot of the time dependence of the amplitude of the damped harmonic oscillator is dependent of the strength of the damping. While the strongest damped case (green curve) has practically vanished, the least damped case (blue curve) has not changed amplitude significantly yet.

Time evolution of damped harmonic oscillator

Normalized Amplitude

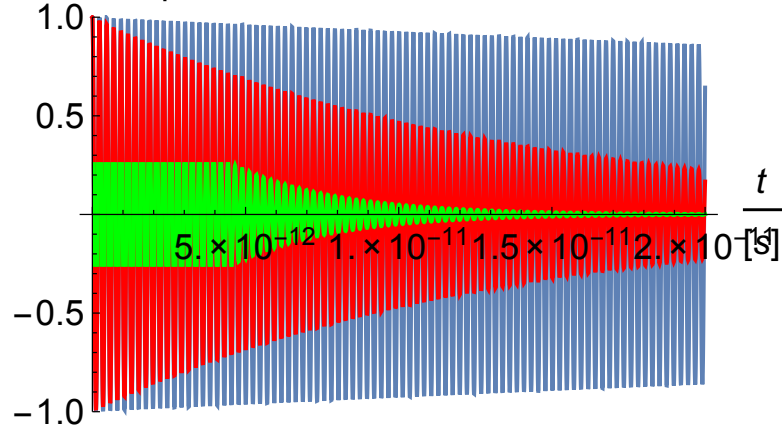


Figure 1: Time dependence of the amplitude of the damped harmonic oscillator for $\Gamma_1 = 0.5\text{cm}^{-1}$ (blue), $\Gamma_2 = 5\text{cm}^{-1}$ (red) and $\Gamma_3 = 20\text{cm}^{-1}$ (green).

- c) Now we consider the inhomogeneous case, i.e., eq. (5): Apply the electric field for times much longer than the decay time so solutions are only of interest for times $t \gg 1/\Gamma \gg 1/\omega_0$ so any homogeneous solution that is initially present will have decayed away (we will expand this later to excitations that last shorter than the decay time). This makes it simple: a particular solution is $x(t) = x_0 e^{-i\omega t}$. Derive x_0 .

Solution:

To derive x_0 we again derive the derivatives of the particular solution again.

$$x(t) = x_0 e^{-i\omega t}$$

$$\dot{x}(t) = x_0 (-i\omega) e^{-i\omega t}$$

$$\ddot{x}(t) = x_0 (-\omega^2) e^{-i\omega t}$$

Then we put these results into the driven differential equation (equation (2)) and get

$$-m x_0 \omega^2 - i m x_0 \Gamma \omega + m x_0 \omega_0^2 = e E_0$$

This yields the following expression for x_0

$$x_0 = \frac{e E_0}{m (\omega_0^2 - \omega^2 - i \Gamma \omega)}$$

- d) The induced optical polarization when the light interacts with a medium can be written as: $p(t) = n e x(t) = \chi \epsilon_0 E(t)$ where χ is called the optical susceptibility of the medium. Using the result from (c), write the susceptibility χ and plot its real and imaginary part for the vibrational oscillator from above. Then use χ to derive an expression for the index of refraction n and extinction coefficient κ and again plot their frequency dependence.

Solution:

Using the given equation we get for χ

$$\chi = \frac{Nex(t)}{\epsilon_0 E(t)}$$

putting in the expressions for $x(t)$ and $E(t)$ we get

$$\chi = \frac{Nex_0}{\epsilon_0 E_0} = \frac{Ne^2}{\epsilon_0 m(\omega_0^2 - \omega^2 - i\Gamma\omega)}$$

To be able to distinguish between real and imaginary part of this expression rewrite χ as follows

$$\chi = \frac{Ne^2}{\epsilon_0 m} \frac{(\omega_0^2 - \omega^2) + i\Gamma\omega}{(\omega_0^2 - \omega^2)^2 + \Gamma^2\omega^2}$$

Thus, the result for the refractive index becomes

$$\tilde{n} = n + i\kappa = \sqrt{1 + \chi}$$

Plotting the real and imaginary part of χ and n gives the following graphs where the blue line always displays the graph for the least damped case and the green line displays the graph for the most damped case assuming an electron density $N = 10^{23} \text{cm}^{-3}$.

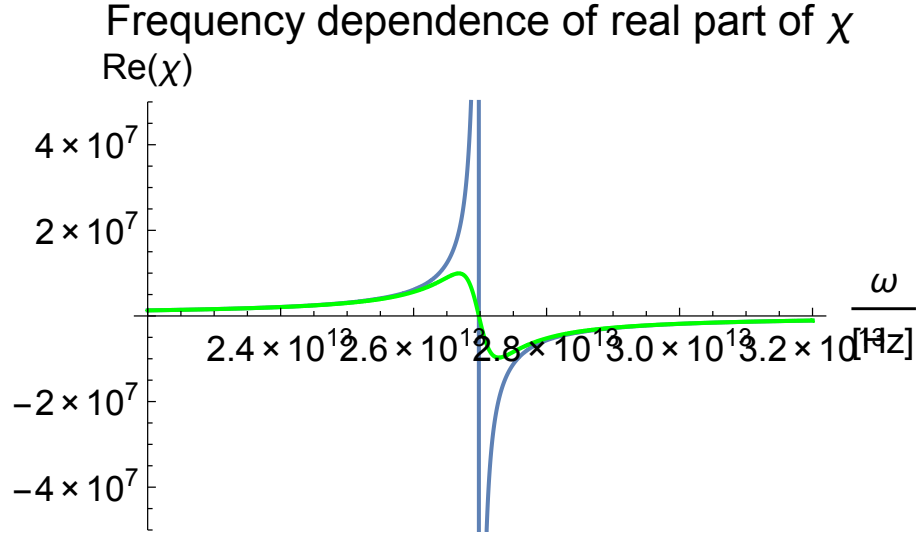


Figure 2: Frequency dependence of the real part of χ for $\Gamma_1 = 0.5 \text{cm}^{-1}$ (blue) and $\Gamma_3 = 20 \text{cm}^{-1}$ (green).

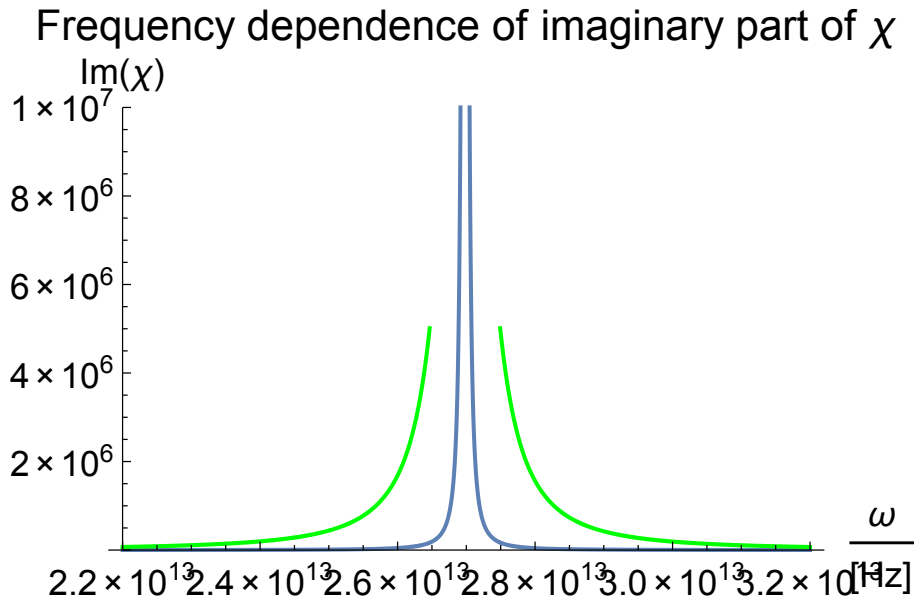


Figure 3: Frequency dependence of the imaginary part of χ for $\Gamma_1 = 0.5cm^{-1}$ (blue) and $\Gamma_3 = 20cm^{-1}$ (green).

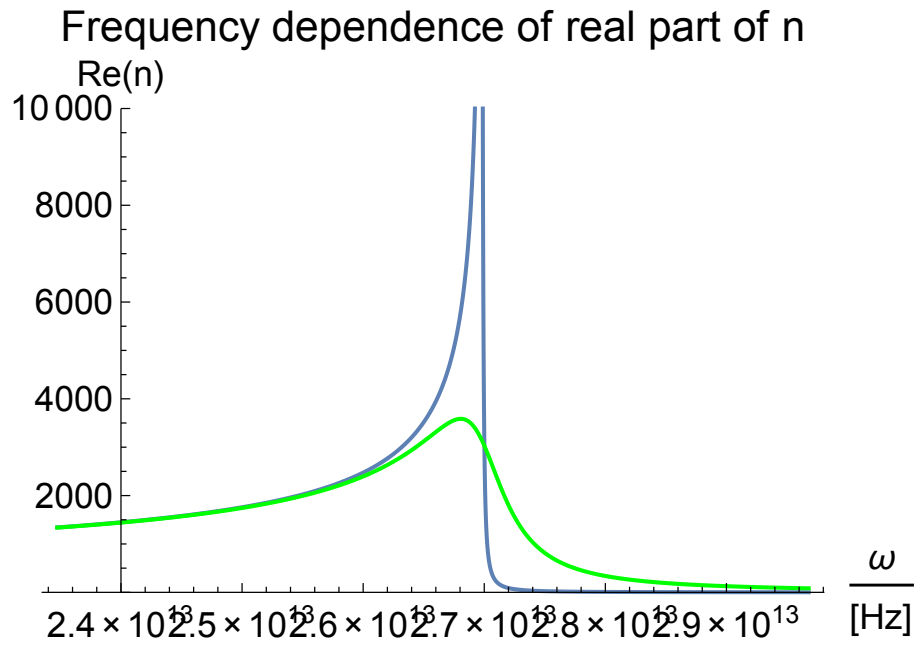


Figure 4: Frequency dependence of the real part of n for $\Gamma_1 = 0.5cm^{-1}$ (blue) and $\Gamma_3 = 20cm^{-1}$ (green).

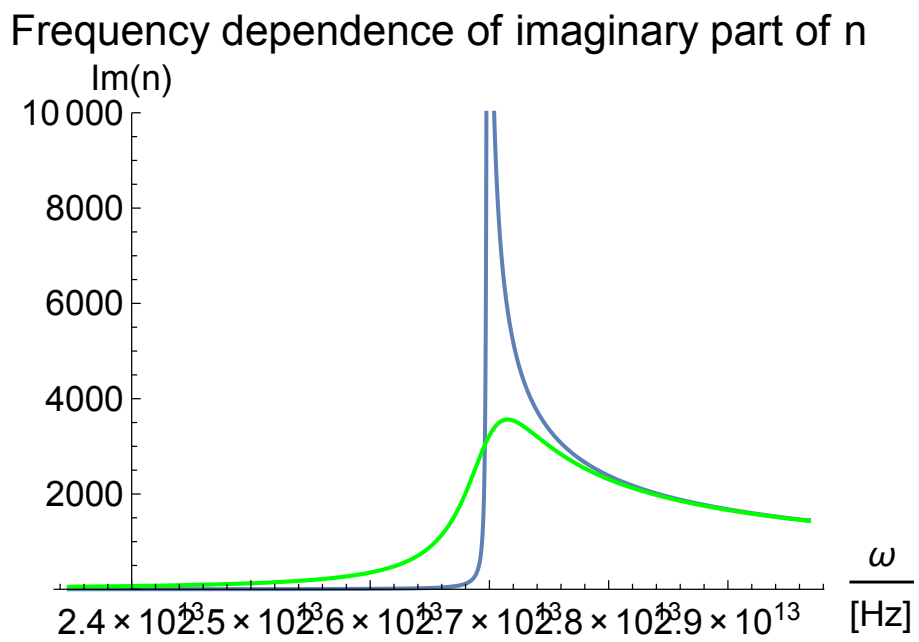


Figure 5: Frequency dependence of the imaginary part of n (κ) for $\Gamma_1 = 0.5\text{cm}^{-1}$ (blue) and $\Gamma_3 = 20\text{cm}^{-1}$ (green).