Error analysis for the physical sciences

A course reader for phys 1140
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Part 1 (chapter 1 – 2)

1.) Precision and Significant Figures

- Precision depends on the measuring device
- Precision determines the number of Significant Figures
- Sig. Fig. Rules:
  - All non-zero numbers count
  - Zeros between non-zero numbers count
  - Zeros that are written simply as place holders do not count
  - Uncertainties have only one Sig. Fig. unless the first digit is a one (1), then carry to two decimal places
- Standard format for stating measurements
- Examples

2.) Error Propagation

- Master Equation
- Specific Cases for Products/Quotients, Sums/Differences, etc…
- Uncertainty in Counting and other special cases
- Justification of addition in quadrature
- Examples

3.) Statistical Analysis

- Mean, Standard Deviation, Standard Deviation of the Mean
- Derivations and Justifications for each
- Standard Deviation of the Mean as the Uncertainty of the Mean
- Examples

4.) Probability

- Gaussian distribution

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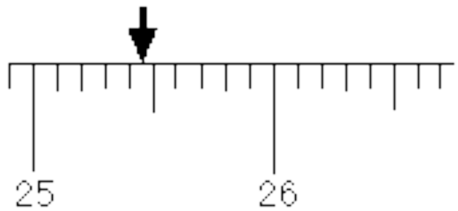
[1] This is the first attempt to write a compact summary of the essential elements of error analysis which the authors hope you find helpful for the phys1140 lab course. We will provide revised versions and subsequent sections within the next few weeks. We would very much appreciate any input you may have with corrections and suggestions for improvements.

1) Precision and Significant Figures:

Any measured physical quantity has an uncertainty. This uncertainty propagates through any form of computation to obtain derived quantities from these measured quantities. A ruler, for example, can only measure lengths to within a millimeter to be *certain* and some fraction of a millimeter is *uncertain*.



The measurement of a distance from zero to the point indicated by the arrow in the figure to the left is 25.45 cm. Because the ruler is divided into centimeters and millimeters, we know for certain that the arrow is between 25.4 and 25.5 cm long, but the last digit is an estimate and thus uncertain due to the limitation of the measuring device. So the measurement should be reported as 25.45 ± 0.05 cm or 254.5 ± 0.5 mm where 0.05 cm or 0.5 mm are the estimated uncertainty in the measurement.

In the measurement of 25.45 cm we have 4 Significant Figures ("Sig Fig's"). We use the number of Sig Fig's to let our reader know the rough estimate of the relative accuracy of the measurement. We only write down as many digits that actually might mean something. So we would not write the above measurement as 25.451873 cm because we cannot say anything beyond the 4th Sig Fig.

Sig. Fig. rules:

1. All non-zero digits (1,2,3,4,5,6,7,8,9) are significant.
2. All zeros between non-zero digits are significant.
3. Zeros that are used purely to set the decimal point (place holders) are generally NOT significant.
4. Trailing zeros that are NOT NEEDED to set the decimal point are significant.

Examples:

X = 3.28	(Three Sig Fig's)	(Rule 1)
Y = 0.0071	(Two Sig Fig's, the zeros in front don't count)	(Rule 1,3)
F = 89.00	(Four Sig Fig's, the zeros after the decimal count)	(Rule 1,4)
G = 2800	(Ambiguous, the zeros may be significant or not)	(Rule 1,3,4)
Q = 2800 ± 10	(Three Sig Fig's, the uncertainty indicates the Sig Fig's)	(Rule 1,4)

Significant Figures give us a rough estimate of the uncertainty in a measurement. More useful, however, is to explicitly report the uncertainty.

"Standard Format" for reporting results

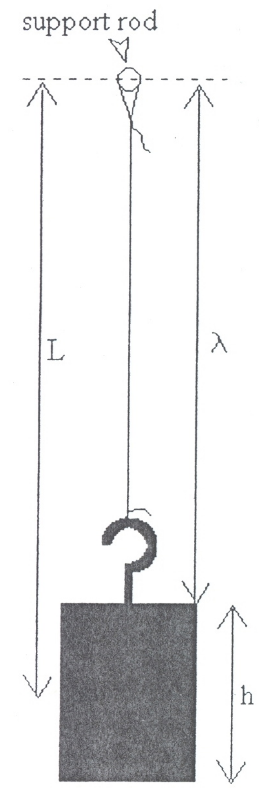
z = (3 8 . 2 ± 0 . 2) cm

- Precision (decimal place) of value and its uncertainty must match.
- As a general rule, use only *one* Sig. Fig. in the uncertainty (possible exception is if the first digit is a "1", then a second digit may be reasonable.)
- Always include units.

D = (6.11864 ± 0.00835) cm WRONG!!! You do not know the uncertainty with that high precision!

D = (6.119 ± 0.008) cm	CORRECT! The uncertainty should have only one Sig Fig and the measurement should be rounded to match the decimal place of the uncertainty.
D = (6.1193 ± 0.0014) cm	CORRECT! The first digit is a 1. Rounding the error to 1 (or rounding up to 2 for values greater than 1.5) may underestimate (overestimate) the error.

Practical example: Measuring the length of a pendulum. (Lab M1)



1. We will begin by first measuring the length λ from the middle of the support rod to the top of the mass.
2. Next we will measure the height of the mass (h).
3. To find the length of the pendulum (L) from the middle of the support rod to the middle of the mass we will add λ + h/2.

The measurements for h and λ have uncertainties due to the limits of our measuring device, in this case a 2-meter ruler. If we were to use a more precise device, such as a laser based measurement system, we may be able to decrease the uncertainty but we would never be able to eliminate it completely.
Due to the uncertainty in our two measurements we will inherently have an uncertainty in the length (L) of our pendulum.

Suppose I make the following measurements:

λ = (50.65 ± 0.05)cm

h = (3.6 0± 0.05)cm

λ = 50.65 ± 0.05 cm	The ± 0.05cm is the uncertainty in our measurement of λ based on the limit of our measuring device. In this case a 2-meter stick. I estimate the 5 at the end of the measurement but I know that the measurement is somewhere between 50.6 and 50.7cm so I make the write the measurement as (50.65 ± 0.05) cm.
h = 3.60 ± 0.05 cm	In the calculation of L, I must consider the estimated uncertainty in both h and λ.

The calculation of L is as follows:

$$L = \lambda + \frac{h}{2} = 50.65 + \frac{3.60}{2} = 52.45 cm$$

This is an easy calculation to do. The uncertainty is a bit more complex. Before we get to the calculation of the uncertainty of L, we must discuss a few rules:

Uncertainty in a Measured Quantity Times an Exact Number.

If the quantity x is measured with uncertainty δx, such that:

$$q = Bx$$

where B has no uncertainty, then the uncertainty in q is:

$$\delta q = |B|\delta x .$$

Uncertainty in Sums and Differences.

Suppose that x, . . . , w are measured values with uncertainties δx, . . . , δw
and the measured values are used to compute:

$$q = x + ... + z - (u + ...w) .$$

If the uncertainties in x, . . . , w are known to be *independent and random* (uncorrelated), then the uncertainty in q is the quadratic sum

$$\delta q = \sqrt{(\delta x)^2 + ... + (\delta z)^2 + (\delta u)^2 + ... + (\delta w)^2}$$

of the individual uncertainties.

We can now calculate the uncertainty in the length L as follows:

$$\delta L = \sqrt{(\delta\lambda)^2 + (\frac{1}{2}\delta h)^2} = \sqrt{(0.05)^2 + (\frac{1}{2}0.05)^2} = 0.0559cm$$

So the final measurement of the length of the pendulum is
L = 52.45 ± 0.06 cm (**Standard Format**)

Notice we rounded the uncertainty to 0.06 so the decimal place in the uncertainty matches the significant figures of the measurement. The calculator will almost always show more decimal places than are appropriate so rounding the calculator answer is almost always done when reporting your measurement in *Standard Format.*

More Examples:
A = 4.39 ± 0.05 m B = 984.0 ± 1.4 m C = 78.1 ± 2.8 m

1. Calculate f = A + C:
   $$A + C = 4.39 + 78.1 = 82.49$$
   $$\delta f = \sqrt{\delta A^2 + \delta C^2} = \sqrt{(0.05)^2 + (2.8)^2} = 2.8004$$

f ± δf = (82 ± 3) m (Standard Format)

2. Calculate A – B

$$f = A - B = 4.39 - 984.0 = -979.61$$

$$\delta f = \sqrt{\delta A^2 + \delta B^2} = \sqrt{(0.05)^2 + (1.4)^2} = 1.40089$$

f ± δf = (-979.6 ± 1.4) m (Standard Format)

Note: In the second example above we round the uncertainty to two decimal places because the first digit in the uncertainty is a one. The calculated measurement is then rounded to 1.4.

2) Error Propagation:

We now continue on the discussion of error propagation which we have already discussed for sums and differences and for the case of a product with an exact number above.

Uncertainties in Products and Quotients.

Suppose that x, . . . , w are measured values with uncertainties δx, . . . , δw
and the measured values are used to compute:

$$q = \frac{x \times \ldots \times z}{u \times \ldots \times w}.$$

If the uncertainties in x, . . . , w are again independent and random (uncorrelated), then the *fractional uncertainty* in q is the sum in quadrature of the corresponding fractional uncertainties for quantity:

$$\frac{\delta q}{|q|} = \sqrt{(\frac{\delta x}{x})^2 + \ldots + (\frac{\delta z}{z})^2 + (\frac{\delta u}{u})^2 + \ldots + (\frac{\delta w}{w})^2}.$$

Example:

Suppose you want to measure the area of a field on which you will plant a crop of wheat. You measure the length of one side of the field to be L = 63.2 ± 0.5 m, and the width to be W = 45.9 ± 0.7 m. What is the calculated area and uncertainty in the area of the field?

$$A = L \times W = 63.2m \times 45.9m$$
$$= 2900.88m^2$$

$$\frac{\delta A}{A} = \sqrt{(\frac{\delta L}{L})^2 + (\frac{\delta W}{W})^2} = \sqrt{(\frac{0.5}{63.2})^2 + (\frac{0.7}{45.9})^2}$$
$$= 0.0172m^2$$

This gives the fractional uncertainty in A. To determine the absolute uncertainty δA, we multiply the calculated value for A by the fractional uncertainty.

$$\delta A = 2900.88m^2 \times 0.0172m^2$$
$$= 49.84m^2$$

The final answer in Standard Format is:
A = 2900 ± 50 m^2 = (2.90 ± 0.5) x 10^3 m^2.

Uncertainty in a Power.

If x is measured with uncertainty δx, and is used to compute:

$$q(x) = x^n$$

where n is a fixed number, then the *fractional uncertainty* is calculated as follows:

$$\frac{\delta q}{q} = |n| \frac{\delta x}{x}.$$

Finally, we discuss the *general case* where we propagate uncertainties through any function of one or more variables.

Uncertainty in any function of one variable.

If x is measured with uncertainty δx, and is used in a function q(x), then the uncertainty in q is calculated by taking the absolute value of the derivative of q with respect to x and multiplying that result by the uncertainty in x. See below:

$$\delta q = \left|\frac{dq}{dx}\right| \delta x.$$

This can be extended for any function of several variables which will give us the:

Uncertainty in a function of several variables.

Suppose that x, . . . , z are measured with uncertainties δx, . . . , δz, and these measurements are used to compute some function f(x, . . . , z). If the uncertainties are all independent and random (uncorrelated) the uncertainty in the function q is found as follows:

$$\boxed{\delta q = \sqrt{(\frac{\partial q}{\partial x}\delta x)^2 + ... + (\frac{\partial q}{\partial z}\delta z)^2}.}$$

THE MASTER RULE!

Where $\frac{\partial q}{\partial \_}$ is the partial derivative of q with respect to each variable used to compute q. This Master Rule always works and any of the special relations (addition/subtraction, products/quotients, etc…) discussed above can be derived from it.

Examples:

1. Suppose we measure the time for a rock to fall into a canyon in order to measure the depth of the canyon. We use a stopwatch and measure the time to be (7.38 ± 0.02) seconds for the time it takes the rock to drop from our hand and reach the bottom of the canyon. What is the depth of the canyon? Include the uncertainty and state the answer in Standard Format.

$$d(t) = \frac{1}{2}gt^2 = \frac{1}{2}(9.81\frac{m}{\sec^2})(7.38\sec)^2$$
$$= 266.876m$$

$$\delta d(t) = \left|\frac{d(d(t))}{dt}\right|\delta t = |gt|\delta t$$
$$= \left|(9.81\frac{m}{\sec^2})(7.38\sec)\right|(0.02\sec) = 1.447m$$

From this information we would write the depth of the canyon in Standard Format as follows:

$d(7.38\sec) = (266.9 \pm 1.4)m$ (exception rule if uncertainty begins with 1)

2. Suppose we want to calculate the acceleration due to gravity. There is an experiment in the lab where we drop a steel marble from a certain height and measure the time of its fall. We use the same equation as above to then determine the acceleration due to gravity. If we measure the distance fallen to be (82.61 ± 0.02) cm and the time to fall that distance to be (0.411 ± 0.001) s, what is the value and uncertainty of the acceleration of gravity?

$$d(t) = \frac{1}{2}gt^2 \Rightarrow g(d,t) = \frac{2d}{t^2}$$
$$g(d,t) = \frac{2(0.8261m)}{(0.411\sec)^2}$$
$$= 9.781\frac{m}{\sec^2}$$

$$\delta g(d,t) = \sqrt{(\frac{\partial g}{\partial d}\delta d)^2 + (\frac{\partial g}{\partial t}\delta t)^2}$$

Before we go any farther we must discuss the idea of *Partial Derivatives*. These are taken when a function depends on more than one variable. In this example the acceleration due to gravity g, is dependent on two variables, namely distance d, and time t. Taking the partial derivative is when we take the derivative of the function with respect to one of the variables, holding everything else constant. In this case:

$$g(d,t) = \frac{2d}{t^2}$$
$$\frac{\partial g}{\partial d} = \frac{2}{t^2}$$
$$\frac{\partial g}{\partial t} = -\frac{4d}{t^3}$$

We take the derivative of g with respect to d (holding time constant) and then we take the derivative with respect to t (holding the distance constant). When we hold the other variables constant, they act as any numerical constant would act when you take the partial derivative.

Applying this knowledge we find for the uncertainty in g:

$$\delta g(d,t) = \sqrt{(\frac{2}{t^2}\delta d)^2 + (-\frac{4d}{t^3}\delta t)^2} = \sqrt{(\frac{2}{.411^2} \times 0.0002)^2 + (-\frac{4 \times 0.8261}{0.411^3} \times 0.001)^2}$$
$$= 0.04759 \frac{m}{\sec^2}$$

Now, as an exercise for your own how would you report the final answer?
It is important to note that the general equation for finding the uncertainty in a function will ALWAYS work for EVERY function. Sometimes it is easier though to simply use the special cases where the function is only addition/subtraction, or multiplication/division, rather than take the partial derivative every time.

More Examples:

A = 10.1 $\pm$ 0.2 m B = 15.91 $\pm$ 0.04 m/sec C = 1.04 $\pm$ 0.02 m/sec D. 2.3 $\pm$ 0.2 sec

Do the following calculations along with the error propagation:

1.

$$v = \frac{A}{D} = \frac{10.1m}{2.3\sec} = 4.39\frac{m}{\sec}$$
$$\delta v = \sqrt{(\frac{\partial v}{\partial A}\delta A)^2 + (\frac{\partial v}{\partial D}\delta D)^2} = \sqrt{(\frac{1}{D}\delta A)^2 + (-\frac{A}{D^2}\delta D)^2} = \sqrt{(\frac{1}{2.3m} * 0.2m)^2 + (-\frac{10.1m}{(2.3\sec)^2} * 0.2\sec)^2}$$
$$\delta v = 0.392\frac{m}{\sec}$$
$$Final: v = 4.4 \pm 0.4\frac{m}{\sec}$$

2.

$$a = \frac{B - C}{D} = \frac{15.91\frac{m}{\sec} - 1.04\frac{m}{\sec}}{2.3\sec}$$

$$a = 6.465\frac{m}{\sec^2}$$

$$\delta a = \sqrt{(\frac{\partial a}{\partial B}\delta B)^2 + (\frac{\partial a}{\partial C}\delta C)^2 + (\frac{\partial a}{\partial D}\delta D)^2} = \sqrt{(\frac{1 - C}{D}\delta B)^2 + (\frac{B - 1}{D}\delta C)^2 + (-\frac{B - C}{D^2}\delta D)^2}$$

$$\delta a = \sqrt{(\frac{1 - 1.04}{2.3} * 0.04)^2 + (\frac{15.91 - 1}{2.3} * 0.02)^2 + (\frac{15.91 - 1.04}{(2.3)^2} * 0.2)^2}$$

$$\delta a = 0.577\frac{m}{\sec^2}$$

$$Final: a = 6.5 \pm 0.6\frac{m}{\sec^2}$$

3.

$$F = \frac{A}{D} + 2(B + C) = \frac{10.1m}{2.3\sec} + 2(15.91\frac{m}{\sec} + 1.04\frac{m}{\sec})$$

$$F = 41.7\frac{m}{\sec}$$

$$\delta F = \sqrt{(\frac{\partial F}{\partial A}\delta A)^2 + (\frac{\partial F}{\partial D}\delta D)^2 + (\frac{\partial F}{\partial B}\delta B)^2 + (\frac{\partial F}{\partial C}\delta C)^2}$$

$$= \sqrt{(\frac{1}{D}\delta A)^2 + (-\frac{A}{D^2}\delta D)^2 + (2\delta B)^2 + (2C\delta C)^2}$$

$$= \sqrt{(\frac{1}{2.3} * 0.2)^2 + (\frac{10.1}{2.3^2} * 0.2)^2 + (2 * 0.04)^2 + (2 * 0.02)^2}$$

$$\delta F = 0.887\frac{m}{\sec}$$

$$Final: F = 41.7 \pm 0.9\frac{m}{\sec}$$

Each of these calculations can be done using the uncertainty equations specifically for multiplication/division, and addition/subtraction and will result in the same uncertainty values as above. You should do this as an exercise.