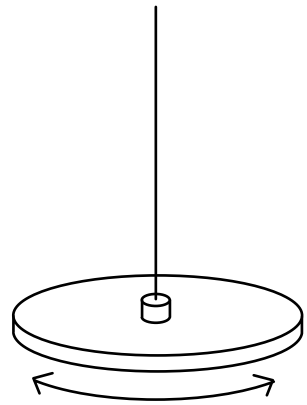
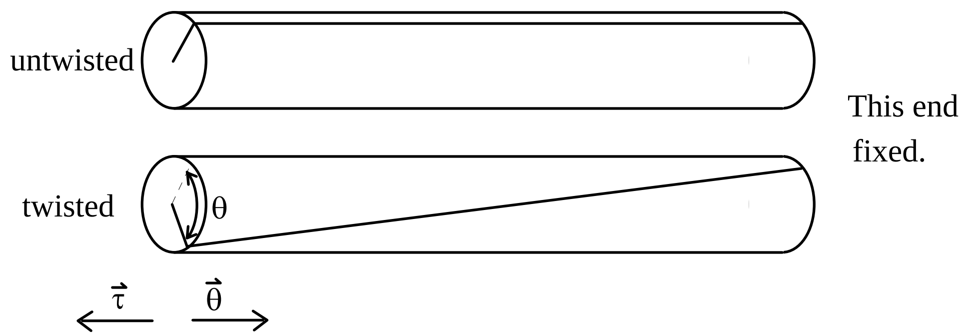
Lab M4: The Torsional Pendulum and Moment of Inertia

Introduction



A torsional pendulum, or torsional oscillator, consists of a disk-like mass suspended from a thin rod or wire. When the mass is twisted about the axis of the wire, the wire exerts a torque on the mass, tending to rotate it back to its original position. If twisted and released, the mass will oscillate back and forth, executing simple harmonic motion. This is the angular version of the bouncing mass hanging from a spring. This lab will give you a better grasp of the meaning of moment of inertia. In Parts 1-3 you will investigate the moment of inertia of disks and rings and the torsional spring constant of a rod. In Part 4, you will use your data to discover how the spring constant depends upon the diameter of the rod.

Consider a thin rod with one end fixed in position and the other end twisted through an angle θ about the rod's axis.



If the angle θ is sufficiently small that the rod is not plastically deformed, the rod exerts a torque τ proportional to the angle θ,

(1) $$\tau = -\kappa\theta \qquad \text{(like } F = -k\,x \text{ for a spring)}$$

where κ (Greek letter kappa) is called the torsion constant. The minus sign indicates that the direction of the torque vector $\vec{\tau}$ is opposite to the angle vector $\vec{\theta}$, so the torque tends to undo the twist. This is just like Hooke's Law for springs.

If a mass with moment of inertia I is attached to the rod, the torque will give the mass an angular acceleration α according to

(2) $$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \qquad \left(\text{like } F = ma = m\frac{d^2x}{dt^2}\right)$$

Combining (1) and (2) yields the equation of motion for the torsional pendulum,

$$I\frac{d^2\theta}{dt^2} = -\kappa\,\theta \qquad \left(\text{like } m\frac{d^2x}{dt^2} = -kx\right)$$

(4) $$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta$$

The solution to this differential equation is

(5) $$\theta(t) = \theta_m\cos(\omega t+\phi) \qquad (\text{like } x(t) = x_m\cos(\omega t+\phi))$$

(6) $$\omega = \sqrt{\frac{\kappa}{I}} \qquad \left(\text{like } \omega = \sqrt{\frac{k}{m}}\right)$$

where θ_m and φ are constants which depend on the initial position and angular velocity of the mass. (The equation of motion is a second order differential equation so its solution must have two constants of integration.) θ_m is the *m*aximum angle; θ oscillates between $+\theta_m$ and $-\theta_m$.

The constant ω is related to the frequency f and the period T of the simple harmonic motion by

(7) $$\omega = 2\pi f = \frac{2\pi}{T}.$$

So, from (6) and (7), the period T is given by

(8) $$T = 2\pi\sqrt{\frac{I}{\kappa}}.$$

It's always good to stare at a formula like this until it makes physical sense. If the wire is very stiff (large κ) then the mass oscillates rapidly and period T is short. If the mass is large so that I is large, then its oscillates slowly and T is long.

The torsion constant can be determined from measurements of T if I is known

(9) $$\kappa = 4\pi^2 \frac{I}{T^2}.$$

Conversely, if κ is known, the moment of inertia I can be determined from measurements of T,

(10) $$I = \kappa \frac{T^2}{4\pi^2}.$$

Experiment

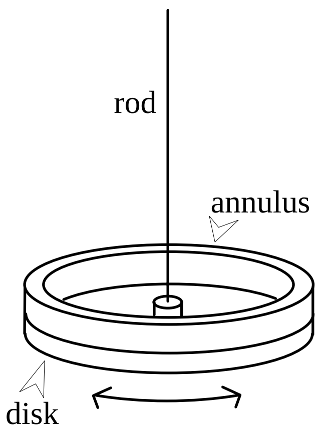
In this first part of this experiment, you will compute the moments of inertia of a disk and an annulus from measurements of the masses and dimensions. In the second part, you will determine the torsion constant of a rod from measurements of T, using a solid disk as a torsion mass. From the definition of moment of inertia I,

(11) $$I = \sum_i m_i r_i^2,$$

it can be shown (you will show it!) that for a solid disk of mass M_{disk} and radius R with the axis of rotation along the symmetry axis,

(12) $$I_{disk} = \tfrac{1}{2} M_{disk} R^2.$$

Thus I_{disk} can be computed from measurements of M and R. Notice that the thickness of the disk does not enter into eq'n (12).



In the third part, you will add to the disk an annular ring with inner radius R_1 and outer radius R_2. If the mass of the annulus is M_{ring}, its moment of inertia is

(13) $$I_{ring} = \tfrac{1}{2} M_{ring} \left(R_1^2 + R_2^2\right).$$

[This formula can be understood qualitatively as follows: From (11), the I of a ring of negligible radial thickness is MR^2. The formula (13) is the average I of two rings, one of radius R_1 and one of radius R_2.]

Finally, in the last part of the experiment, you will determine the torsion constant κ of several wires of different radius r and experimentally discover the functional dependence of κ on r.

Procedure

There are 4 torsion rods used in this experiment. Three of them are made of steel and one is made of brass. To determine which is the brass rod, use a magnet — it will attract the steel rods, but not the brass rod. All the rods have the same length, but the three steel rods have different diameters. The thickest steel rod and the brass rod have the same diameter.

Part 1. Calculation of I_{disk}, I_{ring}

Begin by weighing the solid disk and the annular ring. They are quite heavy (a few kilograms) so use the 6000-gram-capacity scales only. [Do not attempt to weigh them on the smaller capacity scales! They are heavy enough to damage those scales.]

With a meter stick, measure the radius R of the disk and R_1, R_2 of the annulus. (To get the radius, measure the diameter and divide by 2.) Then compute I_{disk} and I_{ring}. For this lab, it is preferable to use cgs units(centimeter-gram-second), rather than mks units, because the values of I in this lab are very small in mks units. As usual, record your computed values for I with the correct number of significant figures.

The disk has a small cylindrical protrusion at its hub which is needed to attach the torsion rod. In one of the pre-lab questions, you will compute the moment of inertia of this little hub. Is its contribution to the total moment of inertia of the disk significant or negligible?

Part 2. Determination of κ of thinnest rod

Choose the thinnest rod and use it to suspend the solid disk from the bracket on the wall as shown on the diagram below. Note that the ends of the rod have indentations where the set screws on the bracket and the disk should seat. Tighten the set screws carefully so you can feel them seat properly.

To prevent the disk from wobbling when it is oscillating, the axis of the disk can be kept fixed with a centering post, consisting of a pointed rod that spins freely on a mount. Place the centering post exactly below the center of the disk and raise the post so that its pointed end is seated in the hole in the center of the disk.

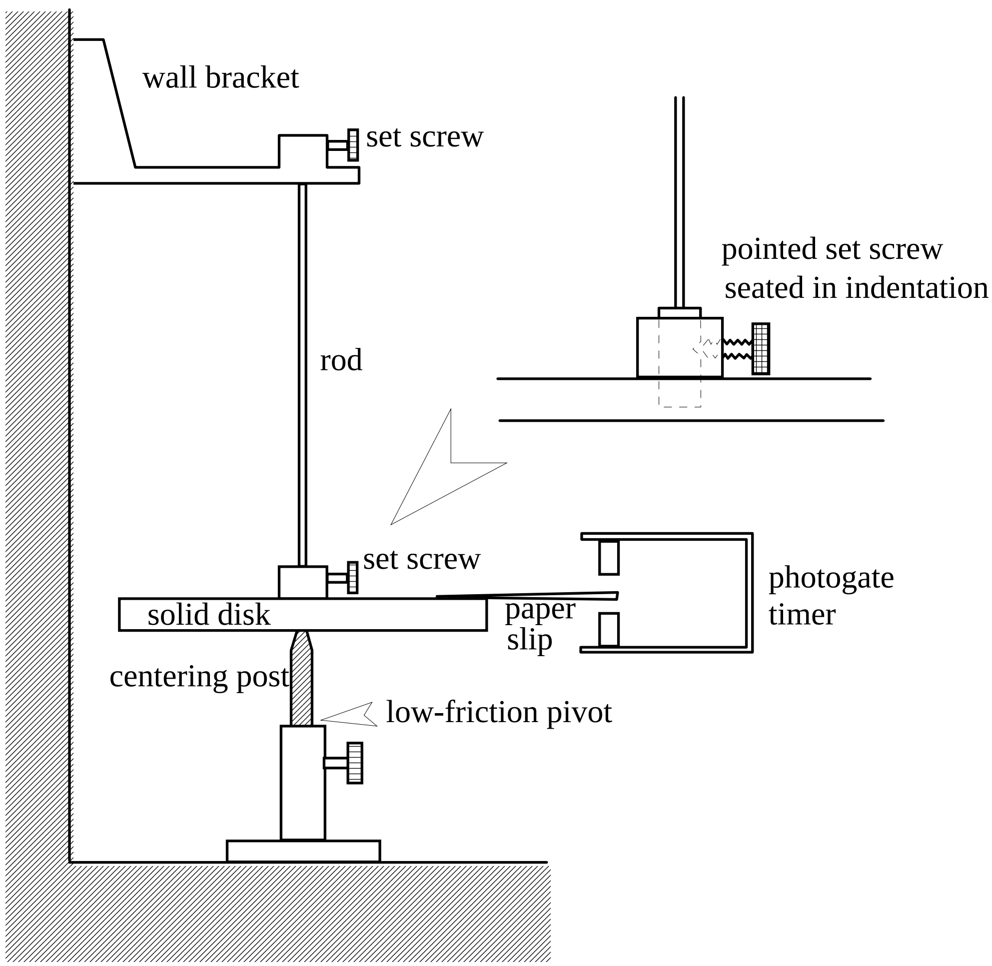
There are two ways to measure the period of the oscillation. You can use a photogate timer in PERIOD mode and tape a piece of paper to the disk so that it interrupts the photogate. Use the reset switch to quickly make several measurements and estimate the best value of T.

An alternative way to measure the period is to use a stop watch and time the interval for several complete oscillations. If the time for N complete periods ($N \approx 20$ or more) is T_{total} , then the period is $T = \frac{T_{total}}{N}$. [DO NOT measure one period twenty times! Measure the time for twenty periods once.] By measuring the time for several periods, the uncertainty in T, due to your reaction time, is reduced by a factor of N.

$$\delta T_{total} = \text{human reaction time} \approx 0.1 \text{ sec.} \qquad \delta T = \frac{\delta T_{total}}{N}$$

For this part of the lab, measure the period both with the photogate timer and with the stopwatch and compare the two results. For subsequent measurements of T, you can choose one or the other.

Now, using your computed I_{disk} and your measured T, compute the torsion constant κ of the thin rod.



Part 3. Measurement of I_{ring}

With both the disk and the annulus suspended from the thin rod, measure the new period T. From your measurements of the period with and without the annulus and your computed I_{disk}, determine I_{ring} , as shown below, and compare with your earlier computed value of I_{ring}.

T_d = period with disk only $\qquad$ T_{rd} = period with disk + ring

$$\kappa = 4\pi^2 \frac{I_d}{T_d^2} = 4\pi^2 \frac{(I_d + I_r)}{T_{rd}^2} \Rightarrow \frac{T_{rd}^2}{T_d^2} = \frac{I_d + I_r}{I_d} = 1 + \frac{I_r}{I_d}$$

$$\Rightarrow I_r = I_d \cdot \left[\left(\frac{T_{rd}}{T_d} \right)^2 - 1 \right]$$

Part 4. Dependence of κ on radius of rod.

Using the precision calipers available (the metal ones in the gray plastic case), determine the radius r of each of the four rods. Using the disk alone (no annulus), measure the period T of the torsion oscillator with each of the three thicker rods. From your measured T's and computed I_{disk}, compute the torsion constant κ for each of the four rods. (κ for the thinnest rod was already determined in part 2.)

For a fixed length rod and a fixed composition, the torsion constant κ varies as some power of the radius r; that is, $\kappa = A\, r^n$, where A and n are constants. Your job is to determine the exponent n. With the data from the three steel rods make a plot of κ vs. r^n where n is some *integer*. Vary n until the graph is a straight line through the origin. Verify that you have the correct power n by computing the ratio $\frac{\kappa}{r^n}$ for the three steel rods. [If the graph is really a straight line through the origin, then the ratio should be a constant for all three rods.]

On the same graph, plot the point for the brass rod. You will have to create a second trace for that single point, and, in the plot format window, set the trace type to points and symbol to box or diamond, etc. Is brass stiffer than steel or vice-versa?

PreLab Questions:

1. What is the *definition* of moment of inertia I? (Write down a general equation for I and explain what the symbols in the equation mean.) What are the units of I in the cgs system(cgs = centimeter-gram-second)? What are the units of I in the MKS system? (MKS = <u>m</u>eter-<u>k</u>ilogram-<u>s</u>econd, also called the SI system.)

2. Use the definition of I to show that the moment of inertia of a thin hoop of mass M, radius R, and negligible thickness is given by $I_{hoop} = MR^2$.

3. Use the definition of moment of inertia to show that $I_{disk} = \frac{1}{2}MR^2$. [Hint: Use the integral form of eq'n (11), $I = \int \rho \; r^2 \, dV$, where $\rho = M / V$ is the density of the disk.]

4. A solid disk of radius R and mass M is glued to a thin hoop of radius R and mass m ($m \neq M$). The symmetry axes of the disk and hoop are aligned. What is I_{total} , the total moment of inertia of the disk + hoop about the axis of symmetry?

5. The disk has a little hub at its center (in the shape of an annulus) with a mass m = 70±1 grams and with inner and outer radii of r_1 = 6.4 mm and r_2 = 16.8 mm. Use eq'n (13) to compute the moment of inertia of the hub. (Careful of your units! Are you going to use kilograms or grams?, meters or centimeters or ...?)

6. What are the units of the torsion constant κ in the MKS system? In the cgs system?

7. What is the *definition* of torque? (Don't just give a vague, qualitative definition; give the precise, technical definition, as given in physics text books. As usual, if you give an equation, explain what the terms in the equation mean.) What are the units of torque in the MKS system?

8. Consider a torsion pendulum consisting of a solid disk of mass M and radius R, suspended from a wire with torsion constant κ Write an equation which shows how the period T depends on M, R, and κ What happens to the period T, if θ_m, the amplitude of the angular motion, is decreased by a factor of two?

9. Consider two disks, labeled A and B, with disk A having two times the mass and three times the radius of disk B ($M_A = 2M_B$, $R_A = 3R_B$). Each disk is suspended from a wire with same torsion constant κ, and the period of each (T_A, T_B) is measured. What is the ratio T_A / T_B ?

10. Consider a set of rods all of the same material and the same length, but with differing radii, r. Assuming that you have the correct value of the exponent n, show with a sketch what a graph of κ vs. r^n should look like.