

## Lab M1: The Simple Pendulum

### Introduction.

The simple pendulum is a favorite introductory exercise because Galileo's experiments on pendulums in the early 1600s are usually regarded as the beginning of experimental physics. Our experiment may be similar to one you have done in high school, however, the mathematical analysis will be more sophisticated. You will need to familiarize yourself (again) with radian measure and trigonometric functions, take derivatives of trigonometric functions in the prelab exercise, and apply the error analysis being presented in lecture. In part I you will find the acceleration of gravity and analyze the error in your measurements. In part II you will take data for your first plot using Mathcad, and in part III your data will illustrate why a small angle is used in part I.

### Theory

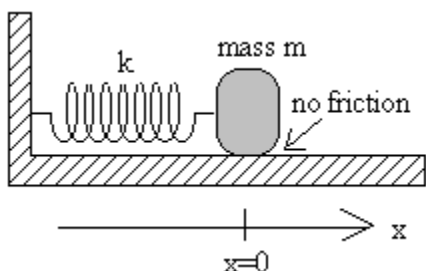
A simple pendulum consists of a mass  $m$  hanging at the end of a string of length  $L$ . The period of a pendulum or any oscillatory motion is the time required for one complete cycle, that is, the time to go back and forth once. If the amplitude of motion of the swinging pendulum is small, then the pendulum behaves *approximately* as a simple harmonic oscillator, and the period  $T$  of the pendulum is given *approximately* by

$$(1) \quad T = 2\pi\sqrt{\frac{L}{g}}$$

where  $g$  is the acceleration of gravity. This expression for  $T$  becomes exact in the limit of zero amplitude motion and is less and less accurate as the amplitude of the motion becomes larger. From this expression, we can use measurements of  $T$  and  $L$  to compute  $g$ .

Let us review simple harmonic motion to see where eq'n (1) comes from. Recall that simple harmonic motion occurs whenever there is a restoring force which is proportional the displacement from equilibrium. The simplest example of simple harmonic motion is that of a mass  $m$  on a spring which obeys Hooke's Law,

$$(2) \quad F_{\text{restore}} = F_{\text{spring}} = -k x$$



where  $x$  is the displacement from equilibrium and  $k$  is a constant called the spring constant. From Newton's Second Law,  $F_{\text{net}} = ma$ , we can

$$\text{write } m \frac{d^2x}{dt^2} = -k x, \text{ or}$$

$$(3) \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad .$$

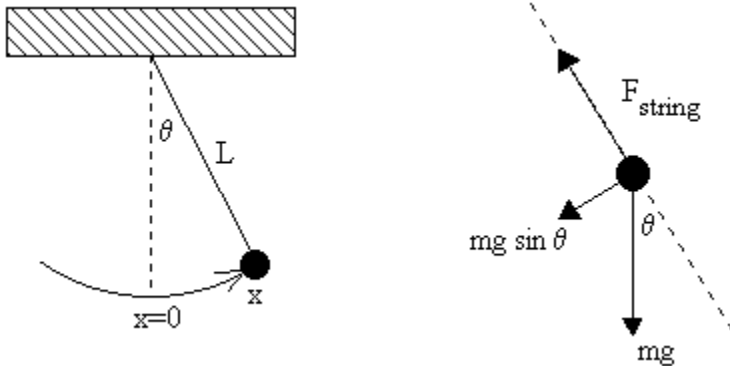
This is a second-order linear differential equation – "second-order" meaning that the equation involves a second derivative, and "linear" meaning that it contains no non-linear  $x$ -terms such as  $x^2$  or  $\sin(x)$ . The solution to equation (3) is

$$(4) \quad x(t) = A \sin(\omega t + \phi), \quad \text{where } \omega = \sqrt{\frac{k}{m}} \quad .$$

The constant  $A$  is the *amplitude* of the motion; the position  $x$  oscillates between  $-A$  and  $+A$ . The constant  $\phi$  is called the phase constant.  $A$  and  $\phi$  depend on the initial conditions, that is, the position and velocity at time  $t=0$ . The period  $T$  is related to  $\omega$  by

$$(5) \quad \omega = \frac{2\pi}{T} \quad .$$

Now consider the simple pendulum — a mass  $m$  hanging on a string of length  $L$ . We can describe the displacement from equilibrium of the mass  $m$  either with the distance along a curved  $x$ -axis or with angle  $\theta$  of the string from the vertical. The distance  $x$  is related to the angle  $\theta$  by  $x = L\theta$ , where  $\theta$  is *in radians*.



There are two forces on the mass  $m$ : the force of gravity,  $mg$ , and the tension in the string,  $F_{\text{string}}$ . The tension  $F_{\text{string}}$  has no component along our curved  $x$ -axis, while gravity has a component along the  $x$ -axis equal to  $-mg \sin \theta$ . The minus sign indicates that the direction (sign) of the force is toward the origin, always opposite to the direction (sign) of  $x$  and  $\theta$ .

If  $\theta$  is very small and expressed in radians, then  $\sin \theta$  is about equal to  $\theta$ , and we have, *approximately*,

$$(6) \quad F_{\text{restore}} \approx -mg\theta = -\frac{mg}{L}x ,$$

and, using  $F = ma$ , we have

$$(7) \quad m \frac{d^2x}{dt^2} = -\frac{mg}{L}x , \quad \frac{d^2x}{dt^2} = -\frac{g}{L}x .$$

In this special case of small  $x$  or  $\theta$ , the restoring force is proportional to the displacement  $x$ , and we have a simple harmonic oscillator whose equation of motion is just like eq'n (3) except we have replaced the constant  $k/m$  with another constant  $g/L$ . The solution to equation (7) is exactly like the solution to eq'n (3) except we replace  $k/m$  with  $g/L$ . So, by looking at eq'n (4), we have, for a simple pendulum

$$(8) \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{L}{g}} .$$

Solving for  $T$  gives us eq'n (1). We emphasize that this expression is true *in the limit* of small amplitude motion (small  $\theta$ , small  $x$ ). Note that, according to eq'n (1), the period is independent of both the mass  $m$  and the amplitude  $A$ .

If we do not assume that  $\theta$  is small so we cannot make the simplifying assumption  $\sin \theta \cong \theta$ , then the restoring force is

$$(9) \quad F_{\text{restore}} = -mg \sin \theta ,$$

and the equation of motion becomes

$$(10) \quad mL \frac{d^2\theta}{dt^2} = -mg \sin \theta , \quad \frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin \theta .$$

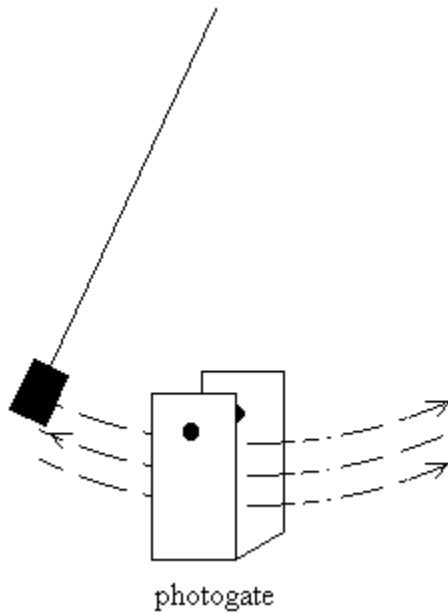
In this case, we have a *non-linear* differential equation, which is quite difficult to solve. In this more complicated case, the period  $T$  still does not depend on  $m$  (since  $m$  does not appear in the equation of motion, it cannot appear in the solution); however, the period  $T$  does depend on the amplitude of motion. We will use the symbol  $\theta_0$  for the amplitude, or maximum value, of  $\theta$ . The exact solution for the period  $T$  can be written as a infinite power series in  $\theta_0$ , and the first three terms of the solution are

$$(11) \quad T = 2\pi \sqrt{\frac{L}{g}} \left( 1 + \frac{1}{16} \theta_0^2 + \frac{11}{3072} \theta_0^4 + \dots \right) .$$

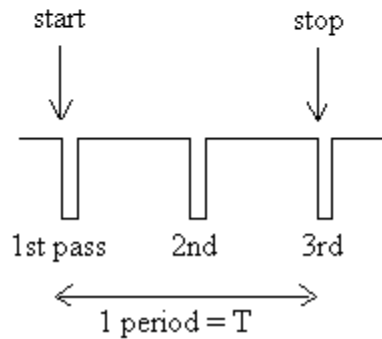
(The terms beyond the first three are quite small and can almost always be ignored.) In this expression, the angle  $\theta_0$  *must be in radians*, not degrees. Note that this expression reduces to eq'n (1) in the limit of small angle  $\theta_0$ .

### **Experimental Procedure.**

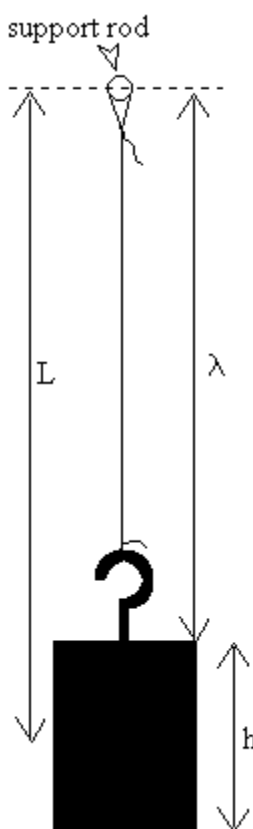
In this lab, we will use a photogate timer to make very precise measurements of the period  $T$  of a simple pendulum as function of the length  $L$  and the amplitude  $\theta_0$ . A photogate consists of an infra-red diode, called the source diode, which emits an invisible infra-red light beam. This beam is detected by another diode, the detector diode. When the mass  $m$  passes between the source and the detector, the infra-red light beam is interrupted, producing an electronic signal that is used to start or stop an electronic timer. In "period mode", the timer starts when the mass first passes through the gate, and does not stop until the mass passes through the gate a third time, as shown below. Thus the timer measures one complete period of the pendulum. Our timers are quite precise and read to  $10^{-4}$ s = 0.1 ms. Pressing the "reset" button sets the timer to zero and readies the timer for another measurement. Check that the timer is working properly by passing a pencil three times through the photogate. The timer should start on the first pass, ignore the second pass, and stop on the third pass.



**Detector signal vs. time**



When making measurements, begin by positioning the photogate so that the mass is directly between the diodes when it hangs straight down. Then, after pulling the mass to the side, release it carefully so that it swings through the photogate without a collision.



Allow the mass to swing back and forth a few times before making measurements to allow for any wobble to settle out. After the mass is set swinging, several measurements of  $T$  can be quickly made by simply pressing the reset button while the mass is still swinging. You should not stop the mass and restart it after each measurement, since this will introduce unnecessary wobble. Always keep an eye on the pendulum while it is swinging so that it does not drift into a collision with the delicate and expensive photogate.

In your pendulum, the string is suspended over a support rod that turns freely on its axis. You will make measurements with 5 or more strings of varying length from about 40 cm to 120 cm. The length  $L$  of the pendulum is the distance from the center of the mass to the pivot point, which is the center of the top axle. To measure  $L$ , first measure the distance  $\lambda$  from the top of the mass to the pivot point; then measure the height  $h$  of the mass.  $L = \lambda + h/2$ . You should only measure  $\lambda$  with the mass suspended, since the weight of the mass stretches the string somewhat.

### Part 1. Precision determination of $g$ .

Choose the longest string and measure the length  $L$  as described above. Since the longest string is longer than 1 meter, use a 2-meter stick. If you are working with a partner, it's a good idea for both of you to measure  $L$  independently and then compare answers. Estimate and record the uncertainty  $\delta L$  in the length  $L$ .

Now measure the period  $T$ . Begin by positioning the photogate carefully and set the mass swinging with a very small amplitude (a few degrees, just enough to pass completely through the photogate each swing). Make three measurements of the period  $T$ . Your "best" value for  $T$  will be the middle of your three measurements. For your uncertainty  $\delta T$ , use half the difference between the highest and lowest measurement.

Compute  $g$  from eq'n (1) using your measured values of  $L$  and  $T$ . Using Mathcad, compute  $\delta g$ , the uncertainty in  $g$ , which is

$$(12) \quad \frac{\delta g}{g} = \sqrt{\left(\frac{\delta L}{L}\right)^2 + \left(2\frac{\delta T}{T}\right)^2}.$$

We don't expect you to understand where this expression comes from just yet. It will be covered in the lectures.

**Report your final answer in the standard format:  $g \pm \delta g$ , with units, being careful not to put more significant figures than you are sure of. (Write the final result using a text box because Mathcad will almost surely give too many significant figures for  $\delta g$ .)**

Calculate  $\Delta g = g - g_{\text{known}}$ , the difference between your answer and  $g_{\text{known}} = 9.796 \text{ m/s}^2$ , a value obtained in a more precise way than we can achieve in the lab. How does this difference  $\Delta g$  compare with the error  $\delta g$  that you calculated? Discuss the largest contribution to the uncertainty in your result and how this might be reduced. Compare your measured  $g$  with  $g_{\text{known}} = 9.796 \text{ m/s}^2$ . If your  $g_{\text{meas}}$  and  $g_{\text{known}}$  do not agree within the uncertainty  $\delta g$ , suggest possible sources of systematic error which might explain the discrepancy.

NOTE: In Mathcad, you cannot define an expression like  $\delta g/g$ ; you can only define a variable like  $\delta g$ . So, your Mathcad document should have a definition like:

$$\delta g := g \cdot \sqrt{\left(\frac{\delta L}{L}\right)^2 + \left(2 \cdot \frac{\delta T}{T}\right)^2}.$$

And remember: all the variables used in the definition ( $g, L, T, \delta L, \delta T$ ) have to be defined previously in the Mathcad document.

Eq'n (12) is the best estimate of the uncertainty due to *random measurement errors* only. It does not take account of any possible *systematic errors*.

## Part 2. Dependence of T on L.

There should be 5 strings with approximate lengths of 40, 60, 80, 100, and 120 cm. If your apparatus is missing some of these lengths, just cut some more string from the ball provided. If your apparatus is not tall enough for 120 cm, pick a different fifth length. Measure the period  $T$  for each of the five lengths. Keep the amplitude of motion very small, so that eq'n (1) is approximately correct. By pressing the reset button several times while the mass is swinging, observe the period for several trials. By looking at the numbers for many trials, *estimate* the average value of the period *and* its uncertainty. (Do **not** record all the trials and do **not** compute the average - that is too time-consuming. We want you to observe and estimate.)

Now we shall see whether your data vary with length in accord with eq'n (1). First, using Mathcad make a plot of  $T$  vs.  $L$  (using small circles or squares for your data) and note that  $T$  does not vary linearly with  $L$ . [Side comment: a graph of "A vs. B" means A on the y-axis and B on the x-axis: Y vs. X, always.]

Squaring both sides of eq'n (1) yields

$$(13) \quad T^2 = \frac{4\pi^2}{g} L .$$

So, if this equation is correct, then a graph of  $T^2$  vs.  $L$  should be a straight line with slope  $4\pi^2/g$ .

Using Mathcad, make a plot of  $(T^2)_i$  vs.  $L_i$  that shows your data as points NOT connected by a line. On the plot of  $T^2$  vs.  $L$ , also plot the straight line  $y(L) = m \cdot L$ , where the slope  $m = 4\pi^2/g$ . Use the known value of  $g = 9.796 \text{ m/s}^2$ . The theory line should be plotted without points to avoid confusion of these points with your data points.

[It is NOT necessary to repeat the error analysis done in part 1 above for each length of string.]

### Part 3. Dependence of T on amplitude.

*Notice that the Mathcad Primer shows, step by step, all the analysis for this section.*

Using the string which is nearest to 1 meter in length, measure the period  $T$  as a function of the maximum angle  $\theta_0$ . Measure the angle  $\theta_0$  using the protractor which is attached to the pendulum rig. Measure  $T$  for several angles, from about  $5^\circ$  up to about  $70^\circ$  or so, at intervals of about  $5^\circ$ . (Avoid angles larger than  $70^\circ$ , simply for safety's sake.) Be *extremely careful* to avoid collisions with the photogate. You need not take multiple trials at each angle; one good measurement at each angle is enough. **Convert your angles from degrees to radians** and then make a graph of  $T(\theta_0)$  vs.  $\theta_0$  (in radians). Don't forget to let  $n = 1, 2, \dots, N$  where you define  $N$ . On the same graph, plot the eq'n (11)

$$(11) \quad T = 2\pi \sqrt{\frac{L}{g}} \left[ 1 + \frac{1}{16} \theta_0^2 + \frac{11}{3072} \theta_0^4 \right],$$

using your measured value of  $L$  and the known value of  $g$ . Theory and experiment should agree pretty closely. Comment on any discrepancy. For example, is the error random (some points too high and some too low) or systematic (all points high or low).

### Pre-lab Questions

**To be handed in at the beginning of your lab session. Please read this lab handout carefully before attempting to answer these questions.**

1. In part 1, when we measure the acceleration of gravity  $g$ , why is it important to keep the amplitude of the swinging pendulum small?
2. What condition or conditions are required for simple harmonic motion to occur?

3. Demonstrate that the solution of eq'n (3) is given by eq'n (4). Do this simply by inserting eq'n (4) into eq'n(3) and checking that an equality is obtained. As an example, consider the differential equation  $\frac{d^2x}{dt^2} = -C \cdot \frac{dx}{dt}$ , where  $C$  is a constant.

The solution of this equation is  $x(t) = Ae^{-Ct} + B$ , where  $A$  and  $B$  are constants. To check that this is indeed the solution, first do the derivatives that appear in the

equation:  $\frac{dx}{dt} = -A \cdot Ce^{-Ct}$  and  $\frac{d^2x}{dt^2} = +A \cdot C^2e^{-Ct}$ . Then plug into the equation to

check for an equality:

$$\frac{d^2x}{dt^2} \stackrel{?}{=} -C \cdot \frac{dx}{dt}$$

$$+A \cdot C^2e^{-Ct} \stackrel{?}{=} -C \cdot (-A \cdot Ce^{-Ct}) \quad (\text{It works!})$$

4. (a) What is the *definition* of the slope of a line? (b) Sketch a graph of  $T^2$  vs.  $L$  for a simple pendulum with a very small amplitude of motion [no numbers on the graph!, just a qualitative sketch] (c) What is the slope of the line in part (b) ? (d) If your graph had numbers with SI units, what would be the units of the slope of the line.
5. Suppose you have a simple pendulum consisting of a mass  $m$  at the end of a massless string of length  $L$ . What would happen to the period  $T$  of the pendulum if you doubled the mass  $m$ ?



6. What is the *definition* of angular measure in radians? Using the definition, show that there are  $\pi$  radians in  $180^\circ$ . [Hint: the statement “ $\pi$  radians =  $180^\circ$ ” is a conversion factor, not a definition. Look up the definition in your physics book.]

7. What is the *fractional increase* in the period  $T$  of a simple pendulum when the maximum angle  $\theta_0$  is increased from an extremely small angle (much less than  $1^\circ$ ) to (a)  $5^\circ$ , (b)  $70^\circ$ ?

[Hint: think about whether to use degrees or radians in your calculations.]

[Note: If  $T$  increases from 1.213 sec to 1.229 sec, the fractional increase is

$$\frac{\Delta T}{T} = \frac{1.229 - 1.213}{1.213} = 0.013. \text{ The } \textit{percentage increase} \text{ is } 1.3\%.]$$

8. Sketch a graph of  $T$  vs. amplitude  $\theta_0$  for a simple pendulum. (No numbers on this graph! Just make a clear, qualitative sketch showing what the graph looks like.)
9. (Counts as 2 questions) In an experiment with a simple pendulum, the following measurements are recorded:  $L = 110.5 \pm 0.4$  cm,  $T = 2.1090 \pm 0.0009$  sec. (a) Based on these measurements, what is the value of  $g$  and what is the value of  $\delta g$ , the uncertainty in  $g$ ? (b) What would be the value of  $\delta g$  if we ignored the small uncertainty in  $T$ ? [Remember to show all your work and to give final answers with the appropriate number of significant figures.] [Hint: in your calculations, should  $L$  be in cm or m?]